Dynamic Stochastic General Equilibrium (DSGE) Models

HSE University

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Outline

- New Keynesian DSGE model: derivations
- Full information estimation methods: Bayesian approach
- Metropolis-Hastings algorithm
- Matlab + Dynare implementation

Dynamic Stochastic General Equilibrium (DSGE) Models

Dynamic

intertemporal vs intratemporal decisions

Stochastic

agents' responses to shocks $\rightarrow$ markets $\rightarrow$ business cycle fluctuations

General Equilibrium

vs partial equilibrium

Strengths

- Microeconomic foundations
- Theoretical coherence
- Quantitative nature
- Response to Lucas critique
- Policy analysis
- Forecasting
- Transparency & openness

Limitations

- Transparency & openness
- Missing forces
- Consistency with evidence
- Financial frictions & crisis

Derivations

Firm's problem:

$$\max_{P_{j,t}} \left\{ \frac{\lambda_t}{P_t} \left[P_{j,t} Y_{j,t} - mc_t Y_{j,t} - \frac{\psi_p}{2} \left(\frac{P_{j,t}}{\Pi_{t-1}^{\iota_p} P_{j,t-1}} - 1 \right)^2 P_t Y_t \right] + \right.$$
$$\left. + \mathbb{E}_t \left[\sum_{s=1}^{\infty} \frac{\lambda_{t+s}}{P_{t+s}} \left[P_{j,t+s} Y_{j,t+s} - mc_{t+s} Y_{j,t+s} - \frac{\psi_p}{2} \left(\frac{P_{j,t+s}}{\Pi_{t+s-1}^{\iota_p} P_{j,t+s-1}} - 1 \right)^2 P_{t+s} Y_{t+s} \right] \right] \right\}$$

Using $Y_{j,t} = \left(\frac{P_{j,t}}{P_t} \right)^{-\varepsilon_{p,t}} Y_t$ and $mc_t = \frac{W_t}{A_t}$, we obtain

Derivations

Using $Y_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon_{p,t}} Y_t$ and $mc_t = \frac{W_t}{A_t}$, we obtain

$$\max_{P_{j,t}} \Bigg\{ \frac{\lambda_t}{P_t} \Bigg[P_{j,t} \left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon_{p,t}} Y_t - \frac{W_t}{A_t} \left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon_{p,t}} Y_t - \\ - \frac{\psi_p}{2} \left(\frac{P_{j,t}}{\Pi_{t-1}^{\iota_p} P_{j,t-1}} - 1 \right)^2 P_t Y_t \Bigg] + \\ + \mathbb{E}_t \Bigg[\sum_{s=1}^{\infty} \frac{\lambda_{t+s}}{P_{t+s}} \Bigg[P_{j,t+s} \left(\frac{P_{j,t+s}}{P_{t+s}}\right)^{-\varepsilon_{p,t+s}} Y_{t+s} - \frac{W_{t+s}}{A_{t+s}} \left(\frac{P_{j,t+s}}{P_{t+s}}\right)^{-\varepsilon_{p,t+s}} Y_{t+s} - \\ - \frac{\psi_p}{2} \left(\frac{P_{j,t+s}}{\Pi_{t+s-1}^{\iota_p} P_{j,t+s-1}} - 1 \right)^2 P_{t+s} Y_{t+s} \Bigg] \Bigg] \Bigg\}$$

Derivations

First Order Condition (FOC):

$$\frac{\lambda_t}{P_t}\left[(1-\varepsilon_{p,t})\left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon_{p,t}} Y_t + \varepsilon_{p,t}\frac{W_t}{A_t}\left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon_{p,t}}\frac{Y_t}{P_{j,t}} - \right.$$
$$\left. - \psi_p\left(\frac{P_{j,t}}{\Pi_{t-1}^{\iota_p} P_{j,t-1}} - 1\right)\frac{P_t Y_t}{\Pi_{t-1}^{\iota_p} P_{j,t-1}}\right] +$$
$$+ \beta\,\psi_p\,\mathbb{E}_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\left(\frac{P_{j,t+1}}{\Pi_t^{\iota_p} P_{j,t}} - 1\right) P_{t+1}\,Y_{t+1}\frac{P_{j,t+1}}{\Pi_t^{\iota_p} P_{j,t}^2}\right] = 0$$

Multiply by $\dfrac{P_t}{\lambda_t Y_t}$ to get

Derivations

Multiply by $\dfrac{P_t}{\lambda_t Y_t}$ to get

$$1-\varepsilon_{p,t}+\varepsilon_{p,t}\frac{W_t}{A_t P_{j,t}}-\psi_p\left(\frac{P_{j,t}}{\Pi_{t-1}^{\iota_p}P_{j,t-1}}-1\right)\frac{P_t}{\Pi_{t-1}^{\iota_p}P_{j,t-1}}+$$
$$+\beta\,\psi_p\,\frac{P_t}{\lambda_t Y_t}\,\mathbb{E}_t\left[\lambda_{t+1}Y_{t+1}\left(\frac{P_{j,t+1}}{\Pi_t^{\iota_p}P_{j,t}}-1\right)\frac{P_{j,t+1}}{\Pi_t^{\iota_p}P_{j,t}^2}\right]=0$$

$$\boxed{1-\varepsilon_{p,t}+\varepsilon_{p,t}mc_t^r-\psi_p\left(\frac{\Pi_t}{\Pi_{t-1}^{\iota_p}}-1\right)\frac{\Pi_t}{\Pi_{t-1}^{\iota_p}}+\beta\,\psi_p\,\frac{1}{\lambda_t Y_t}\,\mathbb{E}_t\left[\lambda_{t+1}Y_{t+1}\left(\frac{\Pi_{t+1}}{\Pi_t^{\iota_p}}-1\right)\frac{\Pi_{t+1}}{\Pi_t^{\iota_p}}\right]=0}$$

Steady State

$$\boxed{1-\varepsilon_{p,t}+\varepsilon_{p,t} mc_t^r-\psi_p\left(\frac{\Pi_t}{\Pi_{t-1}^{\iota_p}}-1\right)\frac{\Pi_t}{\Pi_{t-1}^{\iota_p}}+\beta\,\psi_p\,\frac{1}{\lambda_t Y_t}\,\mathbb{E}_t\left[\lambda_{t+1}Y_{t+1}\left(\frac{\Pi_{t+1}}{\Pi_t^{\iota_p}}-1\right)\frac{\Pi_{t+1}}{\Pi_t^{\iota_p}}\right]=0}$$

Deterministic steady state: such a point that all variables $\rightarrow$ constant, all shocks $\rightarrow$ unconditional means

$$1-\varepsilon_p+\varepsilon_p mc^r-\psi_p\left(\frac{\Pi}{\Pi^{\iota_p}}-1\right)\frac{\Pi}{\Pi^{\iota_p}}+\beta\,\psi_p\,\frac{1}{\lambda Y}\left[\lambda Y\left(\frac{\Pi}{\Pi^{\iota_p}}-1\right)\frac{\Pi}{\Pi^{\iota_p}}\right]=0$$

Steady state:

$$\boxed{mc^r=\frac{\varepsilon_p-1}{\varepsilon_p}}$$

(Log)-linearization

$$f(x) = f(x^*) + \frac{f'(x^*)}{1!}(x - x^*) + \frac{f''(x^*)}{2!}(x - x^*)^2 + \frac{f^{(3)}(x^*)}{3!}(x - x^*)^3 + \cdots$$

$$n! = n(n-1)(n-2)\cdots 1 \qquad 0! = 1$$

$$\hat{x} = x - x^*$$

$$f(x) \approx f(x^*) + f'(x^*)\hat{x}$$

$$f(x, y) \approx f(x^*, y^*) + f_x(x^*, y^*)\hat{x} + f_y(x^*, y^*)\hat{y}$$

$$\tilde{x}_t = \log\left(\frac{x_t}{x^*}\right) = \log\left(1 + \frac{x_t - x^*}{x^*}\right) \approx \frac{x_t - x^*}{x^*}$$

(Log)-linearization

Non-linear difference equation $\mathbb{E}_t x_{t+1} = \psi(x_t)$

1. $x_t = x^* e^{\tilde{x}_t} = x^* e^{\log(x_t/x^*)} = x^* \frac{x_t}{x^*} = x_t$
2. $x^* \mathbb{E}_t e^{\tilde{x}_{t+1}} = \psi\left(x^* e^{\tilde{x}_t}\right)$
3. $x^* e^0 + x^* e^0 \mathbb{E}_t(\tilde{x}_{t+1} - 0) \approx \psi(x^* e^0) + \psi'(x^* e^0) x^* e^0 (\tilde{x}_t - 0)$
4. $x^* + x^* \mathbb{E}_t \tilde{x}_{t+1} = \psi(x^*) + \psi'(x^*) x^* \tilde{x}_t$
5. $\mathbb{E}_t \tilde{x}_{t+1} = \psi'(x^*) \tilde{x}_t$

$$
\begin{aligned}
y_t &= f(x_t, z_t) \\
y^* \tilde{y}_t &= f_x(x^*, z^*) x^* \tilde{x}_t + f_z(x^*, z^*) z^* \tilde{z}_t \\
\tilde{y}_t &= \frac{f_x(x^*, z^*) x^*}{f(x^*, z^*)} \tilde{x}_t + \frac{f_z(x^*, z^*) z^*}{f(x^*, z^*)} \tilde{z}_t \rightarrow \text{Elasticity}
\end{aligned}
$$

(Log)-linearization

$$\boxed{1-\varepsilon_{p,t}+\varepsilon_{p,t}mc_t^r-\psi_p\left(\frac{\Pi_t}{\Pi_{t-1}^{\iota_p}}-1\right)\frac{\Pi_t}{\Pi_{t-1}^{\iota_p}}+\beta\,\psi_p\,\frac{1}{\lambda_t Y_t}\,\mathbb{E}_t\left[\lambda_{t+1}Y_{t+1}\left(\frac{\Pi_{t+1}}{\Pi_t^{\iota_p}}-1\right)\frac{\Pi_{t+1}}{\Pi_t^{\iota_p}}\right]=0}$$

Steady state:

$$\boxed{mc^r=\frac{\varepsilon_p-1}{\varepsilon_p}}$$

Log-linearized:

$$\hat{\lambda}_t^{p,*}=-\frac{\hat{\varepsilon}_{p,t}}{\psi_p}=\frac{\varepsilon_p-1}{\psi_p}\hat{\lambda}_t^p,\ \text{ where } \lambda_t^p=\frac{\varepsilon_{p,t}}{\varepsilon_{p,t}-1}\ \text{ is the time-varying price markup}$$

$$\boxed{\pi_t-\iota_p\pi_{t-1}=\beta\,\mathbb{E}_t[\pi_{t+1}]-\iota_p\pi_t+\frac{\varepsilon_p-1}{\psi_p}\,\widehat{mc}_t^r+\hat{\lambda}_t^{p,*}}$$

Calibration vs Estimation

Limited information

- GMM
- SMM
- Indirect inference
- Penalized GMM

Full information

- Classical
- Bayesian

Bayes' Theorem

Conditional probability

$$p(A \mid B) = \frac{p(A, B)}{p(B)}$$

Joint probability

$$p(A, B) = p(A \mid B)p(B) = p(B \mid A)p(A)$$

Update beliefs

$$p(B \mid A) = \frac{p(A \mid B)p(B)}{p(A)}$$

Applying Bayes' Theorem

Posterior distribution

$$p(\theta \mid y^T) = \frac{L(y^T \mid \theta)p(\theta)}{p(y^T)} \propto L(y^T \mid \theta)p(\theta)$$

Likelihood function

$$L(y^T \mid \theta)$$

Prior distribution

$$p(\theta)$$

Data density

$$p(y^T)$$

Monte Carlo Integration

$$E\big(g(\theta) \mid y^T\big) = \int g(\theta)p(\theta \mid y^T)\, d\theta$$

Posterior distribution

$$\theta_s \sim p(\theta \mid y^T), \qquad s = 1, \ldots, S$$

$$E\big(g(\theta) \mid y^T\big) = \int g(\theta)p(\theta \mid y^T)\, d\theta \approx \frac{1}{S}\sum_{s=1}^{S} g(\theta_s) = \hat{g}_S$$

Priors

Independent priors

$$\theta = (\theta_1, \ldots, \theta_K), \qquad p(\theta) = \prod_{k=1}^{K} p(\theta_k).$$

- Other datasets
- Economic theory
- Economic interpretation
- Multiple peaks
- Facilitate identification
- Prior sensitivity

Priors in Practice

Normal Prior	$(-\infty, +\infty)$	Sign is unknown
Uniform Prior	$[LB, UB]$	Equally likely
Beta Prior	$[0, 1]$	Discount factor
Gamma Prior	$[0, +\infty)$	Feedback parameters
Inverse Gamma Prior	$(0, +\infty)$	Standard deviations

Posterior Sampling

- Importance Sampling
- Gibbs-Sampling
- Metropolis-Hastings algorithm

Metropolis-Hastings Idea

$$p(\theta \mid y^T) \propto L(y^T \mid \theta)p(\theta).$$

Correlated random draws

$$\{\theta_0, \theta_1, \ldots, \theta_N\}.$$

Metropolis-Hastings Algorithm

Proposal density

$$\theta_0, \qquad j = 1, \ldots, N, \qquad \tilde{\theta} \sim q(\,\cdot \mid \theta_{j-1}).$$

Acceptance probability

$$\alpha(\tilde{\theta} \mid \theta_{j-1}) = \min\left\{1, \frac{L(y^T \mid \tilde{\theta}) p(\tilde{\theta}) q(\theta_{j-1} \mid \tilde{\theta})}{L(y^T \mid \theta_{j-1}) p(\theta_{j-1}) q(\tilde{\theta} \mid \theta_{j-1})}\right\}.$$

Accept or reject

$$u \sim U[0,1], \qquad \theta_j = \begin{cases} \tilde{\theta}, & u \leq \alpha(\tilde{\theta} \mid \theta_{j-1}), \\ \theta_{j-1}, & u > \alpha(\tilde{\theta} \mid \theta_{j-1}). \end{cases}$$

Random-Walk Metropolis Hastings

Proposal density

$$q(\tilde{\theta} \mid \theta_{j-1}) = q_{RW}(\tilde{\theta} - \theta_{j-1}).$$

Random variable increment

$$\tilde{\theta} = \theta_{j-1} + z, \qquad z \sim q_{RW}.$$

Speed of convergence

Potential Problems

Transient stage

- Long burn-in
- Burn-in required
- Trace plots

Serial correlation

- Correlated random draws
- Trace plots
- Autocorrelation function

Prior vs Posterior

Informative for updating

$$p(\theta \mid y^T) \neq p(\theta).$$

Two cases

$$p(\theta \mid y^T) \approx p(\theta).$$

Data is uninformative

Data is informative

coincides with $p(\theta)$